NE ORK M L ICAS
A theoretical Mini u and an Open Proble

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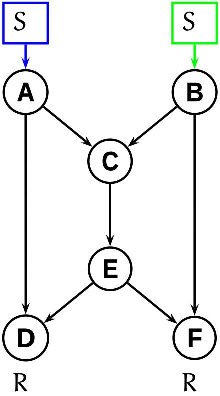
Rutgers

Aalto U., August 2016

NE ORK M L ICAS he Butter y



- **Sources S and S produce bits and .**
- **Each receiver needs bits from both sources.**
- **The edges have unit capacity.**

Can both sources simulaneosly transmit to both reseivers?

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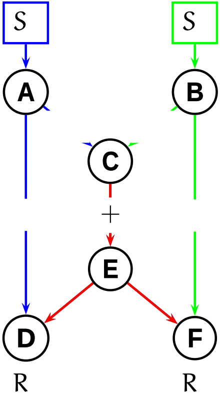

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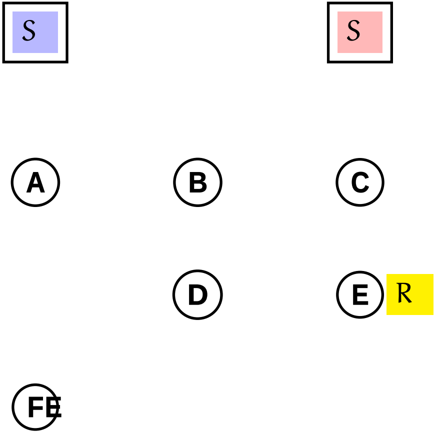
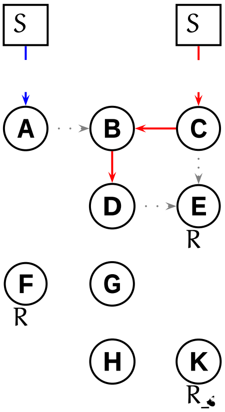
Can both sources simulaneosly transmit to both reseivers?

Yes if nodes can XOR bits.

A Networ for Mu ticast



hree nicasts in a Mu ticast Networ



Networ Multicast heore

Conditions:

- **Network is represented as a**

Networ Multicast heore

Conditions:

- **Network is represented as a directed, acyclic graph.**
- **Edges have unit-capacity and parallel edges are allowed.**
- **There are h unit-rate information sources S , . . . , S .**
- **There are N receivers R , . . . , R located at N distinct nodes.**
- **Between the sources and each receiver node,**
  - **the number of edges in the min-cut is h (or equivalently)**
  - **there are h edge-disjoint paths (S , R) for 1 $\leqslant i \leqslant h$.**

Claim: There exists a multicast transmission scheme of rate h.

Moreover, multicast at rate h

- **cannot always be achieved by routing, but**
- **can be achieved by allowing the nodes to linearly combine their inputs over a su□ciently large finite field.**

NDIREC ED GRAPHS

- The main theorem does not hold.
- Coding can at most double the throughput.

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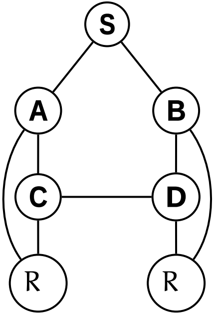

Original Graph

NDIREC ED GRAPHS

-

Networ Multicast Linear Co bining

- Source S, emits , which is an element of some finite field.
- Edges carry **linear combinations of their parent node inputs**.
- Consecutively,
  edges carry **linear combinations of source symbols** , .

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Networ Coding Multicast Prob e

Networ Multicast Exa ple

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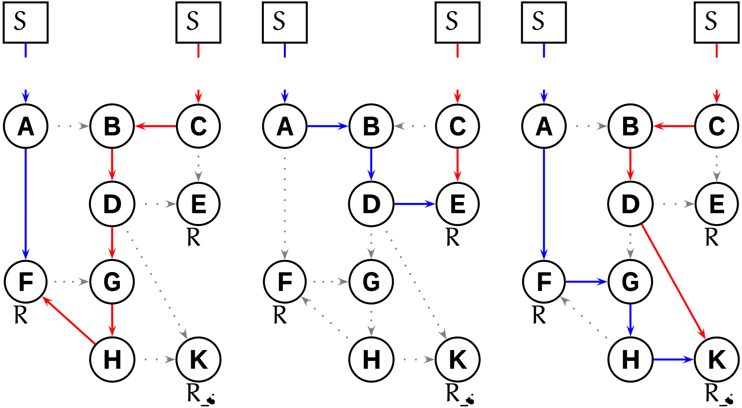

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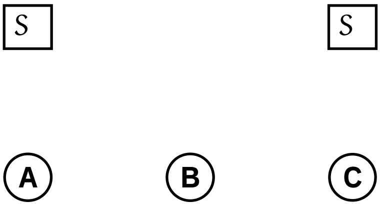

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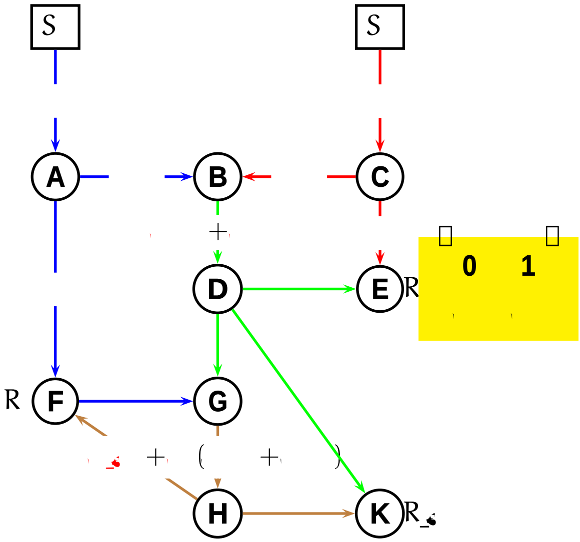

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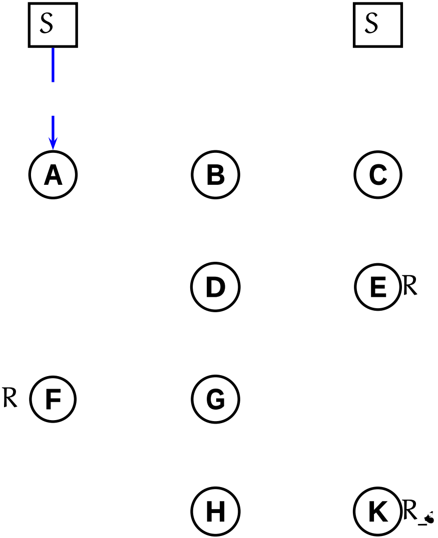

Networ Multicast Code Design

- **Edges carry linear combinations of their parent node inputs; F_v, G are the coefficients used in these linear combinations.**

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- **is the symbol on the last edge of the path (S, R). Receiver j has to solve the following system of equations:**

$$\begin{bmatrix} \\ \vdots \\ \\ \end{bmatrix} = \mathbf{C} \begin{bmatrix} \\ \vdots \\ \\ \end{bmatrix}$$

where the elements of matrix $\mathbf{C}$

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$$\square \quad \square = \mathbf{C} \quad \square \quad \square$$

$$\square\,\vdots\,\square \quad \square\,\vdots\,\square$$

where the elements of matrix $\mathbf{C}$ are polynomials in F G.

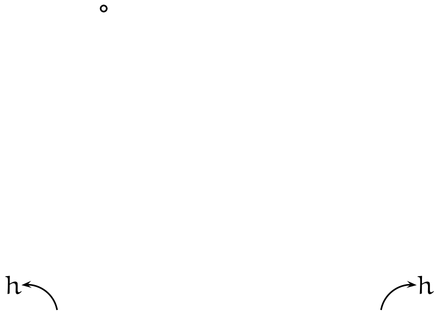
he Code Design Prob e

Networ Mu l ticast Code Existence

Networ Mu|ticast Code Existence

Co bination Networ $B(h\ m)$

A Popular Network With a Small-Alphabt Code



Local and Global Coding Vectors

- **Edges carry linear combinations of their parent node inputs.**
- F G **are the local coding coefficients.**
- **Each edge** e **carries a linear combination of source symbols:**

$$c\ (e) \quad + \cdots + c\ (e) \quad = \begin{bmatrix} c\ (e) \ldots c\ (e) \end{bmatrix} \begin{bmatrix} \\ \vdots \\ \\ \end{bmatrix}$$

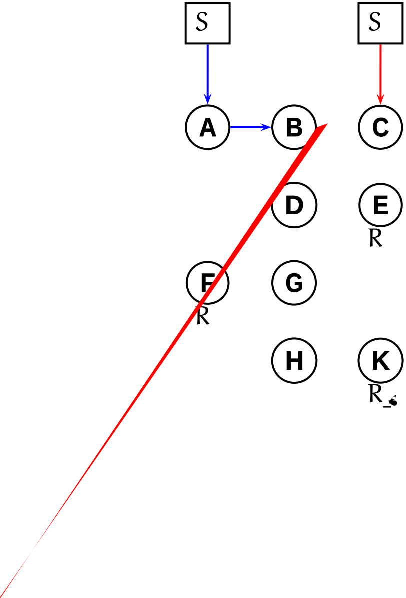
- $[c\ (e) \ldots c\ (e)] \in \mathbb{F}$ **is the global coding vector of edge** e**.**

Decoding for Receiver j

- is the symbol on the last edge on the path (S, R).
- c

Networ Multicast Code Design

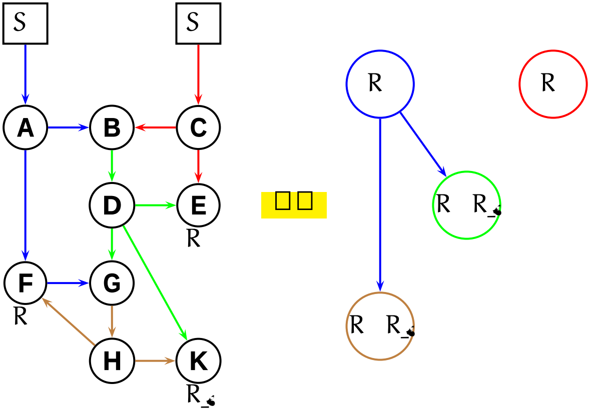
Local and Global iew



Local and Global iew

□

Local and Global iew



Minimal h Multicast Graph ◄ $= (G \ S \ R)$

Ingredients:

1. **Directed, acyclic graph G with**
   - h **source nodes** $S = S \ , \ldots , S$
   - **nodes with in-degree** d, $2 \leqslant d \leqslant h$.
2. **Set of labels** $R = R \ , \ldots , R$ **(receivers).**

Multicast property **(labeling rules):**

1. **Each** R **is used to label exactly** h **nodes. Nodes can have multiple labels.**
2. **Nodes labeled by** R **are connectible to the sources by** h **node-disjoint paths.**

Minimality:

If an edge is removed, the multicast property is lost.

Code Design Problem for Network Multicast

Select a vector in $\mathbb{F}$

he Fie d Size?

heore Fragou i So anin /

Coding for Networ[illegible]s with [illegible]wo Sources

- **Let L be the following set of $(q+1)$ vectors:**

  $[0\ 1]$, $[1\ 0]$, and $[1$ [illegible]$]$ for $0 \leqslant i \leqslant q$

Coding for Networ s with wo Sources

Exa p e

- **Let L be the following set of $(q+1)$ vectors:**

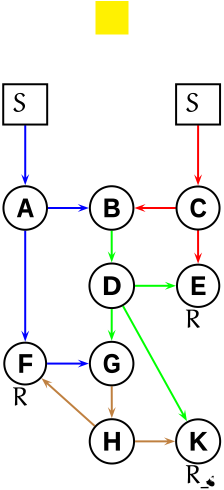
  $$[0\,1],\ [1\,0],\ \text{and}\ [1\ \ ]\ \text{for}\ 0 \leqslant i \leqslant q-2,$$

  where is a primitive element of $\mathbb{F}$.

- **Consider any two di erent vectors in L:**
  - **they are linearly independent, and**
  - **any vector in L is in their linear span.**

=□ **Vectors in L can be treated as colors.**

ertex Co|oring and Code Design



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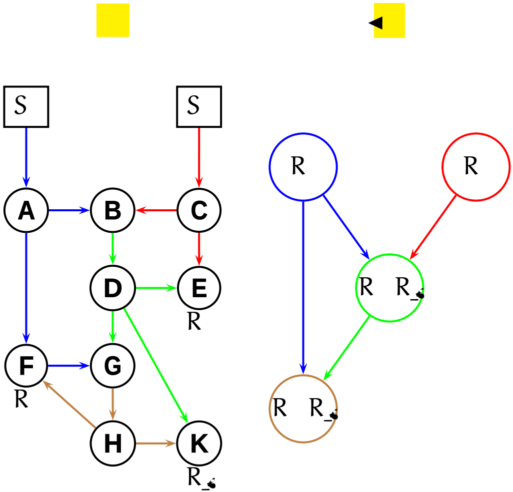

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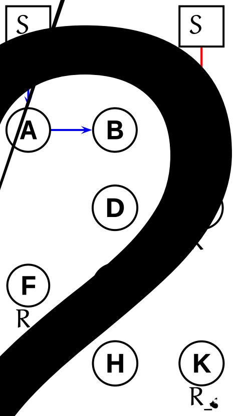

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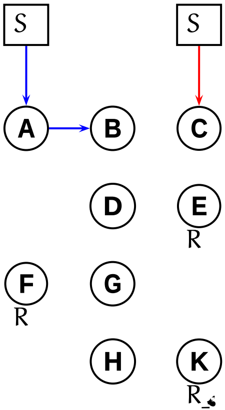

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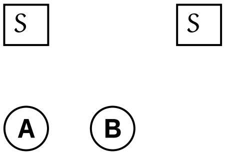

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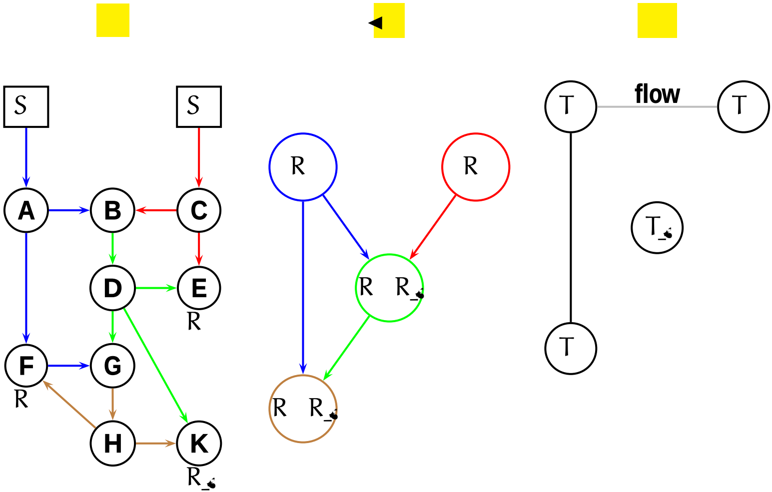

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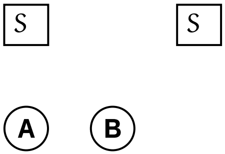

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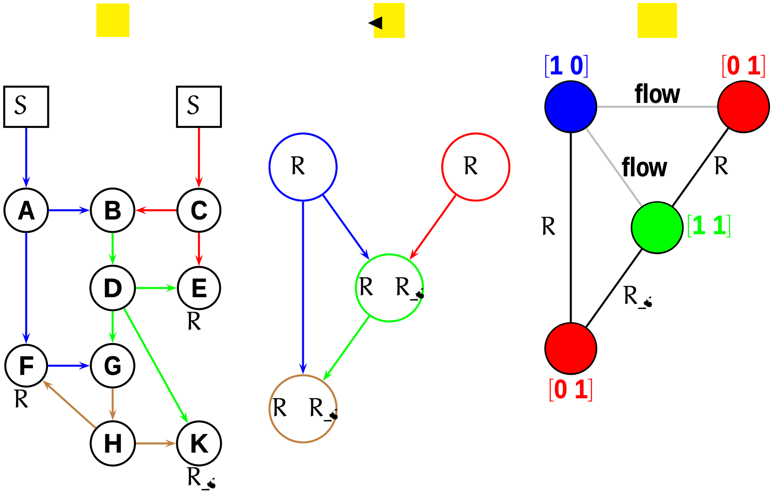


Claim: $\leqslant \sqrt{2N - 7\ 4} + 1\ 2 + 1$

Elements of the Proof:

- **Lemma:** Every vertex in an has degree at least two.

Fie|d Size for Networ with wo Sources

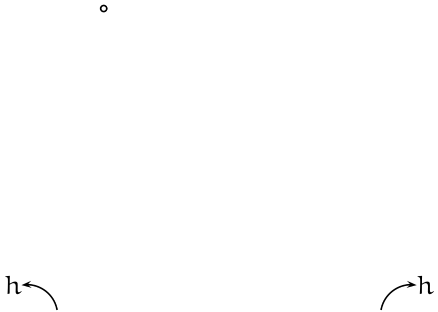
-The Chromatic Number of

Claim:

h

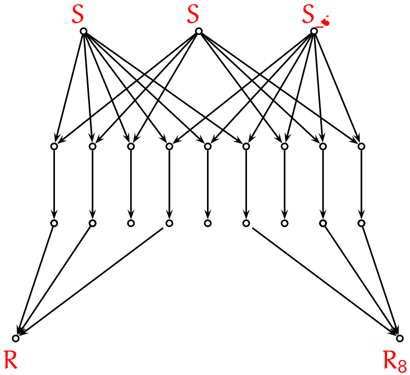
Co bination Networ $B(h\ m)$

A Popular Network With a Small-Alphabt Code



A Distributed Co bination Networ

Fewer than h sources are available at the bottlenecks



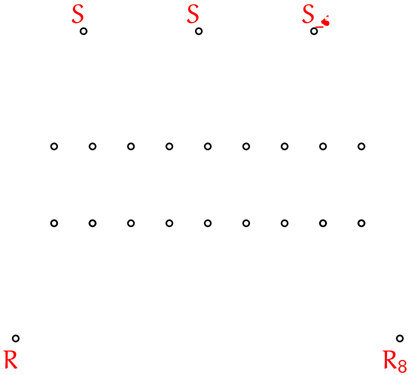
There are

- **3 information sources,**
- **9 bottlenecks, and**
- $\binom{9}{3} - 3 = 81$ **receivers.**

Design a rate-3 multicast!

A Distributed Co bination Networ

Fewer than h sources are available at the bottlenecks



Non Monotonicity

There may be a solution over $\mathbb{F}$ but not over $\mathbb{F}$ for some q 0

Coding vectors for our exa p e networ [illegible]

hat ould e Li e o Do?

... short of solving the problem ...

Find relations (equivalences) with other problems, e.g.,

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Something old :

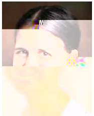
Three problems of Segre in $\mathbb{PG}(h-1, q)$

1. **What is the size $g(h, q)$ of the maximal arc, and which arcs have $g(h, q)$ points?**
2. **For which q and h q are all arcs with $q+1$ points equivalent?**
3. **What are the sizes of the complete arcs, and what is the size of the second largest complete arc?**

Something new :

constrained MDS codes, codes with locality constraints, minimal multicast graph topologies vs. geometry of arcs.

Who are We?



From left to right Fragouli Valdez Manganiello Halbawi Soljanin Anders